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Photon transport in asymmetric random media

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Abstract
The analytical solution for the Stokes vector of a light field inside and outside of a disperse asymmetric isotropic media is obtained. It is assumed that scatterers are contained in a plane-parallel vertically and horizontally homogeneous slab. The diffused light field is studied in the framework of the single-scattering approximation. Equations obtained can be applied for the development of radiative transfer codes for isotropic asymmetric media, based on the adding–doubling procedure.

Keywords: Radiative transfer, turbid media, polarization, optical activity

1. Introduction

Polarized light transport in random media is usually studied in the framework of the vector radiative transfer theory. This theory is based on solutions of the system of integro-differential equations, which can be written in the following form [1, 2]:

$$\frac{\mathrm{d}\vec{S}(\vec{\Omega})}{\mathrm{d}x} = -\hat{\sigma}_{ext}\vec{S}\,(\vec{\Omega}) + \int_{4\pi} \mathrm{d}\vec{\Omega}\,\hat{\sigma}_{sca}(\vec{\Omega}, \vec{\Omega}')\vec{S}(\vec{\Omega}') \qquad (1)$$

where the vector $\vec{\Omega}$ describes the direction of light propagation, $\vec{S}$ is the Stokes vector, $\hat{\sigma}_{ext}$ and $\hat{\sigma}_{sca}$ are extinction and differential scattering matrices and x is the distance. It is assumed here that both medium and light source are stationary and there are no internal sources of radiation inside the medium. Also, the possible change of frequency in the scattering process is neglected. Equation (1) follows from the Maxwell theory, if one allows for several simplifying assumptions [3].

Equation (1) can be applied to disperse media of arbitrary shapes. Experimentalists, however, deal mostly with plane-parallel slabs. Thus, only a transport problem in the plane-parallel geometry is studied here. We assume that a random medium (e.g. a suspension of macroscopic particles) is contained in a plane-parallel slab, which extends to infinity in the horizontal plane. However, a finite thickness along the vertical coordinate axis OZ is assumed (see figure 1). Then it is convenient to measure distances in the direction perpendicular to the border of the medium and $x = z/\cos\vartheta$, where ϑ is the zenith observation angle, counted from the downward vertical, and z is the Cartesian coordinate along axis OZ. The vector $\vec{\Omega}$ in equation (1) in this case is determined by the zenith angle ϑ and azimuth angle ϕ.

We will assume that the medium is uniformly illuminated from the top by a infinitely wide light beam of an arbitrary polarization. The angle of incidence, measured from the vertical, is equal to ϑ_0. Then equation (1) is written in the following form [2]:

$$\cos\vartheta\frac{\mathrm{d}\vec{S}(\vartheta, \phi)}{\mathrm{d}z} = -\hat{\sigma}_{ext}\vec{S}(\vartheta, \phi) + \int_0^{2\pi} \mathrm{d}\phi' \int_0^{\pi} \mathrm{d}\vartheta'\, \sin\vartheta'\hat{\sigma}_{sca}(\vartheta, \vartheta', \phi, \phi')\vec{S}(\vartheta', \phi'). \qquad (2)$$

The task of this paper is to find the solution of equation (2), assuming that secondary and higher-order scattering events can be neglected. Also, we make the assumption of the homogeneity of the random medium under consideration. This means that extinction and scattering matrices are constant throughout the scattering layer studied.

A similar problem was considered by Ishimaru and Yeh [4]. However, they provided the solution only for a light field escaping from the medium. We are interested in the light field inside the slab as well.

We also study the special case of turbid media with optically active spherical particles.

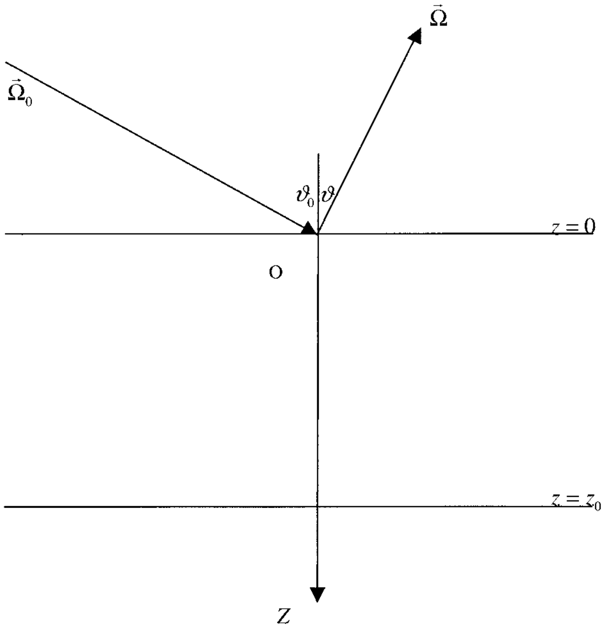

Figure 1. The geometry of the problem.

2. Coherent beam

The Stokes vector $\vec{S}(\vec{\Omega})$ in equation (2) can be presented as the sum of two parts:

$$\vec{S}(\vec{\Omega}) = \vec{I}(\vec{\Omega}) + \vec{I}_c(\vec{\Omega})\delta(\vec{\Omega} - \vec{\Omega}_0) \tag{3}$$

where the Stokes vector $\vec{I}(\vec{\Omega})$ describes the incoherent or diffused light beam and $\vec{I}_c(\vec{\Omega})\delta(\vec{\Omega} - \vec{\Omega}_0)$ is the Stokes vector of the coherent or direct beam propagated in the direction $\vec{\Omega}_0(\vartheta_0, \phi_0)$. Here $\delta(\vec{\Omega} - \vec{\Omega}_0)$ is the delta function.

Let us substitute equation (3) into (2). Then we have two separate equations: one for the diffused light beam Stokes vector $\vec{I}(\vartheta, \phi)$ and one for the direct light beam Stokes vector $\vec{I}_c(\vartheta_0, \phi)$:

$$\cos\vartheta \frac{\mathrm{d}\vec{I}(\vartheta, \phi)}{\mathrm{d}z} = -\hat{\sigma}_{ext}\vec{I}(\vartheta, \phi) + \int_0^{2\pi} \mathrm{d}\phi' \int_0^{\pi} \mathrm{d}\vartheta' \sin\vartheta' \hat{\sigma}_{sca}(\vartheta, \vartheta', \phi, \phi')\vec{I}(\vartheta', \phi') + \hat{\sigma}_{sca}(\vartheta_0, \phi_0, \vartheta, \phi)\vec{I}_c(\vartheta_0, \phi_0) \tag{4}$$

and

$$\cos\vartheta_0 \frac{\mathrm{d}\vec{I}_c(\vartheta_0, \phi_0)}{\mathrm{d}z} = -\hat{\sigma}_{ext}\vec{I}_c(\vartheta_0, \phi_0). \tag{5}$$

Equation (5) can be easily solved [4]. The solution obtained should be substituted into equation (4). In particular, we obtain from equation (5):

$$\vec{I}_c = \hat{T}\vec{J}, \tag{6}$$

where

$$\hat{T} = \exp(-\hat{\sigma}_{ext} l), \tag{7}$$

$l = z/\cos\vartheta_0$ and $\vec{J}$ is the Stokes vector of the incident light.

Equation (4) can be written in the following form to account for equation (6):

$$\cos\vartheta \frac{\mathrm{d}\vec{I}(\vartheta, \phi)}{\mathrm{d}z} = -\hat{\sigma}_{ext}\vec{I}(\vartheta, \phi) + \int_0^{2\pi} \mathrm{d}\phi' \int_0^{\pi} \mathrm{d}\vartheta' \sin\vartheta' \hat{\sigma}_{sca}(\vartheta, \vartheta', \phi, \phi')\vec{I}(\vartheta', \phi') + \hat{\sigma}_{sca}(\vartheta_0, \phi_0, \vartheta, \phi)\hat{T}\vec{J}(\vartheta_0, \phi_0). \tag{8}$$

As far as boundary conditions are concerned we assume that there is no diffused light entering a scattering layer from outside.

Equation (8) is a system of four coupled integro-differential equations. They can be used for studies of polarized diffused light transfer both in symmetric and asymmetric media. For symmetric media, the centre of any large spherical volume is a centre of symmetry and any plane through this centre is a plane of symmetry [5]. This is not the case for asymmetrical media, which usually have some rotatory power. We assume in this study that an asymmetrical medium is macroscopically isotropic, however. The isotropy could in principle be due to an isotropic distribution of small anisotropic elements.

Clearly, equation (8) can be solved using only numerical techniques. A good starting point for the numerical procedure would be the solution of equation (8) in the single scattering approximation [4], when the integral term in equation (8) is neglected. We will consider this approximation in the next section. For this we will need to have the explicit expression for the matrix $\hat{T}$. Let us derive it now.

The derivation can be based on matrix techniques [4, 6] of the solution of the system of ordinary differential equations (5). This is a straightforward way. However, it leads to quite cumbersome final expressions, if we consider arbitrary matrices $\hat{\sigma}_{ext}$ with all non-zero elements. Thus, we use another approach, based on the relationship between the components of the electric field

$$\vec{E} = E_1\vec{e}_1 + E_2\vec{e}_2, \tag{9}$$

where E_1 and E_2 are vectors, which are parallel and perpendicular to the meridional plane defined by the axis OZ and light propagation direction, and the components I_1, I_2, I_3, I_4 of the Stokes vector [2] $\vec{I}$:

$$I_1 = E_1E_1^* + E_2E_2^*, \tag{10}$$

$$I_2 = E_1E_1^* - E_2E_2^*, \tag{11}$$

$$I_3 = E_1E_2^* + E_1^*E_2, \tag{12}$$

$$I_4 = \mathrm{i}(E_1E_2^* - E_1^*E_2), \tag{13}$$

where we have omitted a common multiplier. Quadratic forms (10)–(13) should be averaged, taking into account rapid oscillations of electric field (9) for the finite (and large compared to the period of oscillations) time of optical measurements.

Clearly, the matrix $\hat{T}$ can be obtained directly from the solution [3] of the equation for the coherent field $\vec{E}$:

$$\frac{\mathrm{d}\vec{E}}{\mathrm{d}L} = -\hat{M}\vec{E}, \tag{14}$$

where

$$\hat{M} = 2\pi N k^{-2}\hat{F}(0), \tag{15}$$

N is the number of particles in a unit volume of a random medium, $k = 2\pi/\lambda$, λ is the wavelength and $\hat{F}(0)$ is

the averaged amplitude scattering matrix in the forward direction [2, 4]. The solution of equation (14) has the following form [4]:

$$\vec{E} = \hat{C}\vec{E}_0, \tag{16}$$

where the matrix $\hat{C}$ is as given in appendix A. Let us find the relationship between the matrix $\hat{T}$ in equation (6) and matrix $\hat{C}$ in equation (16).

For this we will use the density matrix [2, 3]

$$\hat{\rho} = \vec{E} \otimes \vec{E}^{+}, \tag{17}$$

where $\otimes$ means the direct product. It follows from equations (16), (17) that

$$\hat{\rho} = \hat{C}\vec{E}_0 \otimes \vec{E}_0^{+}\hat{C}^{+} \tag{18}$$

or

$$\hat{\rho} = \hat{C}\hat{\rho}_0\hat{C}^{+}, \tag{19}$$

where $\hat{\rho}_0 = \vec{E}_0 \otimes \vec{E}_0^{+}$. On the other hand, the Stokes vector components (10)–(13) can be obtained from the following formula:

$$I_j = \mathrm{Tr}(\hat{\sigma_j}\hat{\rho}), \tag{20}$$

where Tr means the trace operation and

$$\hat{\sigma_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{21}$$

$$\hat{\sigma_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{22}$$

$$\hat{\sigma_3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{23}$$

$$\hat{\sigma_4} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{24}$$

Aside from the permutation of indices, these matrices are usually referred to as Pauli spin matrices. These sigma matrices have the following properties:

$$\begin{gathered} \hat{\sigma}_i^{+} = \hat{\sigma}_i, \qquad \hat{\sigma}_i^{2} = \hat{\sigma}_1, \qquad \hat{\sigma}_i\hat{\sigma}_j = \mathrm{i}\hat{\sigma}_k, \\ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{ij}, \qquad \hat{\sigma}_i\hat{\sigma}_j = -\hat{\sigma}_j\hat{\sigma}_i, \end{gathered} \tag{25}$$

where i, j, k take the values 2, 3, 4 in cyclic order and δ_{ij} is the Kronecker symbol.

We obtain from equations (19), (20) for components of the Stokes vector of a coherent part of the light field:

$$(I_c)_j = \mathrm{Tr}(\hat{\sigma}_j\hat{C}\hat{\rho}_0\hat{C}^{+}), \tag{26}$$

or

$$(I_c)_j = \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{k=1}^{4}(\hat{\sigma}_j\hat{C}\hat{\sigma}_k\hat{C}^{+})J_k\right]. \tag{27}$$

where we have used the expansion

$$\hat{\rho}_0 = \tfrac{1}{2}\sum_{k=1}^{4}(\hat{\sigma}_k J_k). \tag{28}$$

The value of J_k is scalar and, therefore, can be taken out of the trace operation. Then it follows from equation (27) that

$$\hat{I}_c = \hat{T}\vec{J}, \tag{29}$$

where elements of the matrix $\hat{T}$ are given by

$$T_{jk} = \tfrac{1}{2}\,\mathrm{Tr}(\hat{\sigma}_j\hat{C}\hat{\sigma}_k\hat{C}^{+}). \tag{30}$$

Equations (29), (30) can be used for studies of coherent beam propagation in a disperse medium. For this one needs to know the matrix $\hat{C}$, which is presented in appendix A.

For instance, it follows for media with optically active spherical particles after simple algebraic calculations that

$$\hat{T} = \begin{pmatrix} \cosh bl & 0 & 0 & -\sinh bl \\ 0 & \cos al & \sin al & 0 \\ 0 & -\sin al & \cos al & 0 \\ -\sinh bl & 0 & 0 & \cosh bl \end{pmatrix} \exp(-\varepsilon l),$$

where $\varepsilon = 4\pi N k^{-2}\,\mathrm{Re}(F_{11}(0))$, $a = 4\pi N k^{-2}\,\mathrm{Re}(F_{12}(0))$, $b = 4\pi N k^{-2}\,\mathrm{Im}(F_{12}(0))$. Here $F_{11}(0) = F_{22}(0)$ and $F_{12}(0) = -F_{21}(0)$ are elements of the amplitude scattering matrix for optically active spherical scatterers [2, 7].

3. Diffused light

3.1. General equations

Let us now consider the diffused or incoherent part of the light field in the framework of the single-scattering approximation. Then the integral term in equation (8), which accounts for multiple light scattering, can be dropped and we arrive at the system of four ordinary nonuniform differential equations:

$$\frac{\mathrm{d}\vec{I}(x)}{\mathrm{d}x} = -\hat{\sigma}_{ext}\vec{I}(x) + \vec{W}(x), \tag{31}$$

where

$$\vec{W}(x) = \hat{\sigma}_{sca}\hat{T}(x)\vec{J} \tag{32}$$

and $x = z/\cos\vartheta$. The elements of the matrix $\hat{T}$ are given by equation (30). The system (31) can be solved analytically by standard techniques [4, 6] (see appendix B). The answer is

$$\vec{I}(x) = \hat{H}\vec{\Upsilon}(x), \tag{33}$$

where the matrix $\hat{H}$ is defined by the following equation:

$$\hat{\varepsilon} = \hat{H}^{-1}\hat{\sigma}_{ext}\hat{H}. \tag{34}$$

Here

$$\hat{\varepsilon} = \begin{pmatrix} \Lambda_1 & 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 & 0 \\ 0 & 0 & \Lambda_3 & 0 \\ 0 & 0 & 0 & \Lambda_4 \end{pmatrix} \tag{35}$$

is the diagonalized extinction matrix. Correspondingly, Λ_1, Λ_2, Λ_3, and Λ_4 are eigenvalues of the extinction matrix $\hat{\sigma}_{ext}$. By definition [6], the columns of the 4×4 matrix $\hat{H}$ are given by the eigenvectors of the extinction matrix $\hat{\sigma}_{ext}$.

The elements of the vector $\vec{\Upsilon}$ in equation (33) are given by (see appendix B)

$$\Upsilon_i(x) = N_{is}\sigma_{sca,sp}F^{i}_{pk}(x)J_k, \tag{36}$$

where $\hat{N} \equiv \hat{H}^{-1}$ and the summation on indices s, p and k (from 1 to 4) is assumed. It follows for the elements of the matrix $\hat{F}^{i}$ (see appendix B) that

$$F^{i}_{pk}(x) = \tfrac{1}{2}\sum_{r=1}^{4}\Psi^{r}_{pk}V^{i}_{r}(x), \tag{37}$$

where

$$\Psi_{pk}^1 = \mathrm{Tr}[\hat{\sigma}_p \hat{D}_1 \hat{\sigma}_k \hat{D}_1^+], \qquad \Psi_{pk}^2 = \mathrm{Tr}[\hat{\sigma}_p \hat{D}_1 \hat{\sigma}_k \hat{D}_2^+], \quad (38)$$

$$\Psi_{pk}^3 = \mathrm{Tr}[\hat{\sigma}_p \hat{D}_2 \hat{\sigma}_k \hat{D}_1^+], \qquad \Psi_{pk}^4 = \mathrm{Tr}[\hat{\sigma}_p \hat{D}_2 \hat{\sigma}_k \hat{D}_2^+] \quad (39)$$

and

$$V_r^i(x) = \frac{\exp[-\Lambda_r ux] - \exp[-\Lambda_i x + (\Lambda_i - \Lambda_r u)\alpha]}{\Lambda_i - \Lambda_r u}, \quad (40)$$

where

$$u = \frac{\cos\vartheta}{\cos\vartheta_0}. \quad (41)$$

Matrices $\hat{D}_1$ and $\hat{D}_2$ are given in appendix A. It is assumed that $\alpha = 0$ for the radiation propagated downwards ($\vartheta \leqslant \frac{\pi}{2}$) and $\alpha = z_0 \sec\vartheta$ (z_0 is the geometrical thickness of the layer) for the radiation propagated upwards ($\vartheta > \frac{\pi}{2}$). Note that the value of u is negative for the radiation propagated upwards and positive otherwise.

We have from equation (40) for the reflected light ($\vartheta > \frac{\pi}{2}$) at the upper boundary ($x = 0, \alpha = z_0 \sec\vartheta$) of a layer that

$$V_r^i(0) = \frac{1 - \exp[(\Lambda_i - \Lambda_r u) z_0 \sec\vartheta]}{\Lambda_i - \Lambda_r u}. \quad (42)$$

It follows for the transmitted light at the bottom of a slab ($x = \ell, \alpha = 0$) that

$$V_r^i(\ell) = \frac{\exp[-\Lambda_r u\ell] - \exp[-\Lambda_i \ell]}{\Lambda_i - \Lambda_r u}, \quad (43)$$

where $\ell = z_0 \sec\vartheta$.

Summarizing, we underline that for the calculation of the Stokes vector of the diffused singly scattered light in a given plane-parallel slab we need to find first of all the eigenvalues and eigenvectors of the extinction matrix in this medium. This can be easily done, using a variety of analytical and numerical approaches [6]. The second step is the calculation of traces (38), (39). Finally, the Stokes vector is calculated using equations (33), (36), (37).

3.2. Asymmetric isotropic media with spherical particles

Let us demonstrate the procedure for the special case of media with optically active spherical particles. The relation of extinction and scattering matrices of media with such particles to their refractive indices and sizes is well known [7]. In particular, the extinction matrix in this case simplifies to the following general form [2]:

$$\hat{\sigma}_{ext} = \begin{pmatrix} \varepsilon & 0 & 0 & -b \\ 0 & \varepsilon & a & 0 \\ 0 & -a & \varepsilon & 0 \\ -b & 0 & 0 & \varepsilon \end{pmatrix} \quad (44)$$

with $\Lambda_1 = \varepsilon - b$, $\Lambda_2 = \varepsilon + \mathrm{i}a$, $\Lambda_3 = \varepsilon - \mathrm{i}a$, $\Lambda_4 = \varepsilon + b$ and

$$\hat{H} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Let us find traces (38), (39) now. First of all we note that 2×2 matrices $\hat{D}_1$ and $\hat{D}_2$ have the following simplified forms for the case under consideration (see appendix A):

$$\hat{D}_1 = \hat{D}_1^+ = \frac{\hat{\sigma}_1 + \hat{\sigma}_4}{2}, \qquad \hat{D}_2 = \hat{D}_2^+ = \frac{\hat{\sigma}_1 - \hat{\sigma}_4}{2}. \quad (45)$$

The evaluation of traces (see equations (38), (39)) gives

$$\Psi_{11}^1 = \Psi_{11}^4 = \Psi_{22}^2 = \Psi_{22}^3 = \Psi_{33}^2 = \Psi_{33}^3 = \Psi_{44}^1 = \Psi_{44}^4 = \Psi_{14}^1 = -\Psi_{14}^4 = \Psi_{41}^1 = -\Psi_{41}^4 = 1 \quad (46)$$

and

$$-\Psi_{23}^2 = \Psi_{23}^3 = \Psi_{32}^2 = -\Psi_{32}^3 = \mathrm{i}. \quad (47)$$

Other values of Ψ_{pk}^r are equal to zero in the case under study.

Then it follows from equation (37) that

$$\hat{F}^i = \frac{1}{2}\begin{pmatrix} V_1^i + V_4^i & 0 & 0 & V_1^i - V_4^i \\ 0 & V_2^i + V_3^i & -\mathrm{i}(V_2^i - V_3^i) & 0 \\ 0 & \mathrm{i}(V_2^i - V_3^i) & V_2^i + V_3^i & 0 \\ V_1^i - V_4^i & 0 & 0 & V_1^i + V_4^i \end{pmatrix}. \quad (48)$$

Thus, the problem of finding the Stokes vector of the diffused light in an asymmetric turbid medium is reduced to the simple matrix multiplications (see equations (33), (36)).

3.3. Symmetric isotropic media

The assumption of a symmetric medium gives $\hat{F} = V\hat{E}$, where $\hat{E}$ is the unity matrix and (see equation (40))

$$V(x) = \frac{\exp[-\Lambda ux] - \exp[-\Lambda x + (1-u)\Lambda\alpha]}{\Lambda(1-u)}, \quad (49)$$

where Λ is the extinction coefficient. We have accounted for the fact that $\hat{\varepsilon} = \Lambda\hat{E}$ for the symmetric media. Then we have

$$\vec{I} = V\hat{\sigma}_{sca}\vec{J}, \quad (50)$$

which is a familiar result [8].

Specifically, we have for the reflected light at the upper boundary ($x = 0, \alpha = \ell$):

$$V^{ref} = \frac{1 - \exp\left[-\Lambda z_0\left(\frac{1}{\cos\vartheta_0} - \frac{1}{\cos\vartheta}\right)\right]}{(1-u)\Lambda}. \quad (51)$$

Also, we obtain for the transmitted diffused light at the bottom of a slab ($L = \ell, \alpha = 0$):

$$V^{tr} = \frac{\exp\left[-\frac{\Lambda z_0}{\cos\vartheta_0}\right] - \exp\left[-\frac{\Lambda z_0}{\cos\vartheta}\right]}{(1-u)\Lambda}. \quad (52)$$

Indices *ref* and *tr* indicate reflected or transmitted light correspondingly. The angles ϑ and ϑ_0 are counted starting from the positive direction of the axis OZ (downward vertical). Thus, the value of $\mu_0 = \cos\vartheta_0$ is always positive for the radiation entering a layer from the top. The same applies to the value of $\cos\vartheta$ in the case of transmitted radiation. However, the value of $\cos\vartheta$ is always negative for the reflected light. Thus, introducing $\mu = |\cos\vartheta|$, we obtain

$$V^{ref} = \frac{1 - \exp\left[-\Lambda z_0\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right]}{(\mu_0 + \mu)\Lambda}\mu_0, \quad (53)$$

$$V^{tr} = \frac{\exp\left[-\frac{\Lambda z_0}{\mu_0}\right] - \exp\left[-\frac{\Lambda z_0}{\mu}\right]}{(\mu_0 - \mu)\Lambda}\mu_0. \quad (54)$$

Combining equations (50), (53), we obtain for the reflected light

$$\vec{I} = \frac{1 - \exp\left[-\Lambda z_0\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right]}{(\mu_0 + \mu)\Lambda}\mu_0\hat{\sigma}_{sca}\vec{J}. \quad (55)$$

A similar equation is easily obtained for the transmitted light.

Let us introduce the phase matrix

$$\hat{P} = \frac{\hat{\sigma}_{sca}}{4\pi\sigma_s}, \tag{56}$$

where σ_s is the total scattering coefficient, and the reflection vector function

$$\vec{\Re} = \frac{\vec{I}}{\mu_0 F}, \tag{57}$$

where $\mu_0 \pi F$ is the incident light flux density. Then it follows from equations (55), (57) that

$$\vec{\Re} = \frac{\omega_0 \hat{P}}{4(\mu_0 + \mu)} \left\{ 1 - \exp\left[-\left[\frac{1}{\mu} + \frac{1}{\mu_0} \right] \tau \right] \right\} \vec{J}_0, \tag{58}$$

where $\vec{J}_0 = \vec{J}/\pi F$, $\tau = \Lambda z_0$ is the optical thickness and $\omega_0 = \sigma_s/\Lambda$ is the single scattering albedo. Equation (58) was reported by, for example, Hansen and Travis [8]. This confirms our derivations.

4. Conclusions

The problem of polarized coherent and incoherent light propagation in symmetric and asymmetric media was considered. The results obtained are based on the solution of the vector radiative transfer equation in the framework of the single-scattering approximation. Formulae derived (see equations (29), (30), and (33), (36), (37)) allow for a simple calculation of the Stokes vector of coherent and singly scattered diffused light in symmetric and asymmetric isotropic disperse media with arbitrary differential scattering and extinction matrices. The power of the general method presented was demonstrated for the specific case of spherical optically active particles. We also re-derived equation (58), which is often used for the case of symmetric isotropic media [3, 8].

Results reported can be used for the interpretation of light scattering experiments, dealing with optically active light scattering media [9–12].

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Appendix A. The matrix $\hat{C}$

The matrix $\hat{C}$ is expressed through elements of the matrix $\hat{M}$ (see equation (15)) by the following equation [4]:

$$\hat{C} = \hat{D}_1 \,\mathrm{e}^{-\gamma_1 l} + \hat{D}_2 \,\mathrm{e}^{-\gamma_2 l}, \tag{A.1}$$

where

$$\begin{aligned} \hat{D}_1 &= \frac{1}{1 - A_{12}A_{21}} \begin{pmatrix} 1 & -A_{12} \\ A_{21} & -A_{12}A_{21} \end{pmatrix}, \\ \hat{D}_2 &= \frac{1}{1 - A_{12}A_{21}} \begin{pmatrix} -A_{12}A_{21} & A_{12} \\ -A_{21} & 1 \end{pmatrix}, \end{aligned} \tag{A.2}$$

and

$$A_{12} = \frac{M_{12}}{\gamma_2 - M_{11}}, \qquad A_{21} = \frac{M_{21}}{\gamma_1 - M_{22}}, \tag{A.3}$$

$$\gamma_1 = \tfrac{1}{2}\left(M_{11} + M_{22} + \sqrt{4M_{12}M_{21} + (M_{11} - M_{22})^2}\right), \tag{A.4}$$

$$\gamma_2 = \tfrac{1}{2}\left(M_{11} + M_{22} - \sqrt{4M_{12}M_{21} + (M_{11} - M_{22})^2}\right). \tag{A.5}$$

The value of l is the length of the beam path. It follows in the case of a plane-parallel layer that $l = z \sec\vartheta_0$, where z is the geometrical depth and ϑ_0 is the incidence angle.

The matrix T in equation (30) can be written in the following form, using equation (A.1):

$$T_{jk} = \tfrac{1}{2}\sum_{r=1}^{4} \Psi_{jk}^r v^r, \tag{A.6}$$

where

$$\Psi_{jk}^1 = \mathrm{Tr}[\hat{\sigma}_j \hat{D}_1 \hat{\sigma}_k \hat{D}_1^+], \qquad \Psi_{jk}^2 = \mathrm{Tr}[\hat{\sigma}_j \hat{D}_1 \hat{\sigma}_k \hat{D}_2^+], \tag{A.7}$$

$$\Psi_{jk}^3 = \mathrm{Tr}[\hat{\sigma}_j \hat{D}_2 \hat{\sigma}_k \hat{D}_1^+], \qquad \Psi_{jk}^4 = \mathrm{Tr}[\hat{\sigma}_j \hat{D}_2 \hat{\sigma}_k \hat{D}_2^+] \tag{A.8}$$

and

$$v_r = \exp(-\Lambda_r z \sec\vartheta_0). \tag{A.9}$$

Here we have used the fact [4] that

$$\begin{aligned} \Lambda_1 &= \gamma_1 + \gamma_1^*, & \Lambda_2 &= \gamma_1 + \gamma_2^*, \\ \Lambda_3 &= \gamma_1^* + \gamma_2, & \Lambda_4 &= \gamma_2 + \gamma_2^*. \end{aligned} \tag{A.10}$$

It follows for optically active spherical particles [7] that $M_{11} = M_{22}$, $M_{12} = -M_{21}$ and, therefore,

$$\gamma_1 = M_{11} + \mathrm{i}M_{12}, \qquad \gamma_2 = M_{11} - \mathrm{i}M_{12}, \tag{A.11}$$

$$A_{12} = A_{21} = \mathrm{i}, \tag{A.12}$$

$$\hat{D}_1 = \hat{D}_1^+ = \frac{\hat{\sigma}_1 + \hat{\sigma}_4}{2}, \qquad \hat{D}_2 = \hat{D}_2^+ = \frac{\hat{\sigma}_1 - \hat{\sigma}_4}{2}. \tag{A.13}$$

Thus, we have for the electric vector $\vec{E}$ of a coherent part of wave propagating in random media with optically active spherical particles (see equation (16)) that

$$\begin{aligned} \vec{E} = {} & \tfrac{1}{2}\{(\hat{\sigma}_1 + \hat{\sigma}_4)\exp[-(M_{11} + \mathrm{i}M_{12})l] \\ & + (\hat{\sigma}_1 - \hat{\sigma}_4)\exp[-(M_{11} - \mathrm{i}M_{12})l]\}\vec{E}_0. \end{aligned} \tag{A.14}$$

Here $\vec{E}_0$ is the electric vector of an incident wave.

Appendix B. The solution of the system of ordinary differential equations

Let us solve equation (31). For this we multiply equation (31) by the matrix $\hat{H}^{-1}$:

$$\hat{H}^{-1}\frac{\mathrm{d}\vec{I}}{\mathrm{d}x} = -\hat{H}^{-1}\hat{\sigma}_{ext}\vec{I} + \hat{H}^{-1}\vec{W} \tag{B.1}$$

and introduce a new unknown vector:

$$\vec{Y} = \hat{H}^{-1}\vec{I}. \tag{B.2}$$

Then it follows that

$$\frac{\mathrm{d}\vec{Y}}{\mathrm{d}x} = -\hat{\varepsilon}\vec{Y} + \vec{\beta}, \tag{B.3}$$

where (see equation (35))

$$\hat{\varepsilon} = \hat{H}^{-1}\hat{\sigma}_{ext}\hat{H} \tag{B.4}$$

and $\vec{\beta} = \hat{N}\vec{W}$, $\hat{N} = \hat{H}^{-1}$. Recall that $\vec{W} = \hat{\sigma}_{sca}\hat{T}\vec{J}$ (see equation (32)). Then it follows that $\vec{\beta} = \hat{N}\hat{\sigma}_{sca}\hat{T}\vec{J}$.

Thus, we should solve four independent ordinary differential equations (B.3), which can be easily done by standard techniques. The answer is

$$Y_i(x) = \int_{\alpha}^{x} \exp(-\Lambda_i(x-s))\beta_i(s)\,\mathrm{d}s, \tag{B.5}$$

where (see appendix A)

$$\beta_i(s) = N_{is}\sigma_{sca,sq}T_{qp}(s)J_p, \tag{B.6}$$

$$T_{qp}(s) = \sum_{r=1}^{4}\Psi_{qp}^{r}v_r(s), \tag{B.7}$$

$$v_r(s) = \exp(-\Lambda_m us), \tag{B.8}$$

$i = 1, 2, 3, 4$, $u = \frac{\cos\vartheta}{\cos\vartheta_0}$, $\alpha = 0$ for the radiation propagated downwards ($\vartheta \leqslant \frac{\pi}{2}$) and $\alpha = z_0 \sec\vartheta$ (z_0 is the geometrical thickness of the layer) for the radiation propagated upwards ($\vartheta > \frac{\pi}{2}$). Here and below, the summation on repeating indices is assumed. We used equation (30) and accounted also for boundary conditions:

$$Y_i(0) = 0 \qquad \text{at } u > 0$$

$$Y_i(\ell) = 0 \qquad \text{at } u < 0,$$

where $\ell = z_0 \sec\vartheta$.

Substituting equation (B.6) into (B.5), we obtain

$$Y_i(x) = N_{is}\sigma_{sca,sq}F_{qp}^{i}(x)J_p, \tag{B.9}$$

where

$$F_{qp}^{i}(x) = \frac{1}{2}\sum_{r=1}^{4}\Psi_{qp}^{r}V_r^{i}(x), \tag{B.10}$$

and

$$V_r^{i}(x) = \frac{\exp[-\Lambda_r ux] - \exp[-\Lambda_i x + (\Lambda_i - \Lambda_r u)\alpha]}{\Lambda_i - \Lambda_r u}. \tag{B.11}$$

Finally, the Stokes vector is obtained from equation (B.2):

$$\vec{I}(x) = \hat{H}\vec{Y}(x) \tag{B.12}$$

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